

# ON SOME PROPERTIES OF PARTIAL SUMS

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**ABSTRACT:** The paper gives some criteria for partial sums of rational number sequences to be not rational functions and to be not algebraic functions. As an application, we study partial sums of some famous rational number sequences in mathematical analysis with relevance to number theory.

## 1 Introduction

To determine properties of partial sums  $S(n) = \sum_{i=1}^n u_i$  of number sequences  $(u_i)_{i \geq 1}$  is usually an interesting problem. Although one has given different tools for this problem, but it is hard to know that functions for partial sums are algebraic functions or transcendental functions.

In this paper, we give some criteria for functions in one positive integer variable to be not rational functions (see Theorem 2.1) and to be not algebraic functions (see Theorem 2.5). In particular, we get the following corollaries for partial sums of rational number sequences (see Section 2). Let  $(u_i)_{i \geq 1}$  be a rational number sequence. Then  $S(n) = \sum_{i=1}^n u_i$  is not a rational function in  $n$  if any one of the following conditions holds (see Corollary 2.3):

- (a)  $\lim_{n \rightarrow \infty} S(n) = \infty$  and  $\lim_{n \rightarrow \infty} \frac{S(n)}{n} = 0$ .
- (b)  $\lim_{n \rightarrow \infty} S(n) = 0$  and  $\lim_{n \rightarrow \infty} nS(n) = \infty$ .
- (c)  $\lim_{n \rightarrow \infty} S(n)$  is an irrational number.

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*Mathematics Subject Classification(2010):* Primary 11B83. Secondary 12B31; 11B34.

*Key words and phrases:* Partial sums; number sequences; transcendental function.

And if  $\lim_{n \rightarrow \infty} S(n)$  is a transcendental number then  $S(n) = \sum_{i=1}^n u_i$  is a transcendental function in  $n$  (see Corollary 2.8).

Using these criteria, we study partial sums of some famous rational number sequences in mathematical analysis with relevance to number theory (see Section 3).

## 2 Some Criteria for Partial Sums

This section gives some criteria for functions in one positive integer variable to be not rational functions and algebraic functions.

Denote by  $\mathbb{Q}; \mathbb{C}$  the field of the rational numbers; the field of the complex numbers, respectively. Let  $K$  be a subfield of  $\mathbb{C}$ ;  $\alpha \in \mathbb{C}$ . Let  $f(n)$  be a complex-valued function in one positive integer variable  $n$ . Then one can make the following definitions:

- (i)  $\alpha$  is a *rational number* if  $\alpha \in \mathbb{Q}[i]$ . And  $\alpha$  is an *irrational number* if  $\alpha \notin \mathbb{Q}[i]$ .
- (ii)  $\alpha$  is *algebraic* over  $K$  if there exist  $c_0, \dots, c_h \in K$  with  $c_h \neq 0$  such that

$$c_h \alpha^h + c_{h-1} \alpha^{h-1} + \dots + c_1 \alpha + c_0 = 0.$$

And  $\alpha$  is a *transcendental number* over  $K$  if it is not an algebraic number over  $K$ .

- (iii)  $\alpha$  is an *algebraic number* if  $\alpha$  is algebraic over  $\mathbb{Q}$ . And  $\alpha$  is a *transcendental number* if it is not an algebraic number. Denote by  $\overline{\mathbb{Q}}$  the field of the algebraic numbers.
- (iv)  $f(n)$  is a *rational function* over  $K$  if

$$f(n) = \frac{a_m n^m + a_{m-1} n^{m-1} + \dots + a_1 n + a_0}{b_d n^d + b_{d-1} n^{d-1} + \dots + b_1 n + b_0}$$

where  $a_0, \dots, a_m; b_0, \dots, b_d \in K$  with  $b_d \neq 0$ .

- (v)  $f(n)$  is an *algebraic function* over  $K$  if there exist polynomials in  $n$  with coefficients in  $K$ :  $q_0(n), \dots, q_k(n) \in K[n]$  with  $q_k(n) \neq 0$  such that

$$q_k(n)[f(n)]^k + q_{k-1}(n)[f(n)]^{k-1} + \dots + q_1(n)[f(n)] + q_0(n) = 0.$$

And  $f(n)$  is a *transcendental function* over  $K$  if it is not an algebraic function over  $K$ .

Then we obtain the following results.

**Theorem 2.1.** *Let  $f(n)$  be a function in one positive integer variable  $n$ . Then the following statements hold.*

- (i) *If  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 0$  then  $f(n)$  is not a rational function in  $n$  over  $\mathbb{C}$ .*
- (ii) *If  $\lim_{n \rightarrow \infty} f(n) = 0$  and  $\lim_{n \rightarrow \infty} nf(n) = \infty$  then  $f(n)$  is not a rational function in  $n$  over  $\mathbb{C}$ .*
- (iii) *If  $\lim_{n \rightarrow \infty} f(n)$  is an irrational number then  $f(n)$  is not a rational function in  $n$  over  $\mathbb{Q}[i]$ .*

*Proof.* The proof of (i): If  $f(n)$  is a rational function over  $\mathbb{C}$  then  $f(n) = \frac{P(n)}{Q(n)}$  here  $P(n)$  and  $Q(n)$  are two polynomials in  $n$  over  $\mathbb{C}$ . Then on the one hand, since  $\lim_{n \rightarrow \infty} f(n) = \infty$ , it follows that  $\deg P(n) > \deg Q(n)$ . On the other hand, since  $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = \lim_{n \rightarrow \infty} \frac{P(n)}{nQ(n)} = 0$ ,  $\deg nQ(n) > \deg P(n)$ . From the above facts we get

$$\deg P(n) \geq \deg Q(n) + 1 = \deg nQ(n) > \deg P(n),$$

contradiction. Arguing similarly with the proof of (i), we obtain (ii). The proof of (iii): Suppose that  $f(n)$  is a rational function over  $\mathbb{Q}[i]$ . Since  $\lim_{n \rightarrow \infty} f(n) = c$  is an irrational number,  $c \neq 0$ . Consequently, we have

$$f(n) = \frac{a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0}{b_m n^m + b_{m-1} n^{m-1} + \cdots + b_1 n + b_0}$$

where  $a_0, \dots, a_m; b_0, \dots, b_m \in \mathbb{Q}[i]$  with  $a_m b_m \neq 0$ . Note that

$$\lim_{n \rightarrow \infty} \frac{a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0}{b_m n^m + b_{m-1} n^{m-1} + \cdots + b_1 n + b_0} = \frac{a_m}{b_m} \in \mathbb{Q}[i].$$

So  $c \in \mathbb{Q}[i]$ . This contradiction shows that  $f(n)$  is not a rational function in  $n$  over  $\mathbb{Q}[i]$ .  $\square$

**Remark 2.2.** The reverse of Theorem 2.1 does not hold. For (i):  $f(n) = n^{\frac{4}{3}}$  is not a rational function and  $\lim_{n \rightarrow \infty} f(n) = \infty$ , but  $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = \infty \neq 0$ . For (ii):  $g(n) = n^{-\frac{5}{4}}$  is not a rational function for which  $\lim_{n \rightarrow \infty} g(n) = 0$ , but  $\lim_{n \rightarrow \infty} ng(n) = 0$ . For (iii):  $q(n) = 2^{-n}$  is not a rational function, but  $\lim_{n \rightarrow \infty} q(n) = 0 \in \mathbb{Q}$ .

Let  $(u_i)_{i \geq 1}$  be a rational number sequence. Then  $S(n) = \sum_{i=1}^n u_i$  is a function in  $n$ . Hence from Theorem 2.1 we immediately get the following corollary.

**Corollary 2.3.** *Let  $(u_i)_{i \geq 1}$  be a rational number sequence. Then  $S(n)$  is not a rational function in  $n$  over  $\mathbb{Q}[i]$  if any one of the following conditions holds:*

- (i)  $\lim_{n \rightarrow \infty} S(n) = \infty$  and  $\lim_{n \rightarrow \infty} \frac{S(n)}{n} = 0$ .
- (ii)  $\lim_{n \rightarrow \infty} S(n) = 0$  and  $\lim_{n \rightarrow \infty} nS(n) = \infty$ .
- (iii)  $\lim_{n \rightarrow \infty} S(n)$  is an irrational number.

**Remark 2.4.** In general, the reverse of Corollary 2.3 does not hold. Indeed, for (i): Choose  $(u_i = i.i!)_{i \geq 1}$ , then

$$S(n) = \sum_{i=1}^n i.i! = (n+1)! - 1$$

is not a rational function and  $\lim_{n \rightarrow \infty} S(n) = \infty$ , but  $\lim_{n \rightarrow \infty} \frac{S(n)}{n} = \infty \neq 0$ . For (ii): Let  $(u_i)_{i \geq 1}$  be a rational number sequence defined by  $u_1 = -\frac{1}{2}$  and  $u_i = \frac{i}{(i+1)!}$  for all  $i \geq 2$ . Then  $S(n) = \sum_{i=1}^n \frac{i}{(i+1)!} = -\frac{1}{(n+1)!}$  is not a rational function for which  $\lim_{n \rightarrow \infty} S(n) = 0$ , but  $\lim_{n \rightarrow \infty} nS(n) = 0 \neq \infty$ . For (iii):  $S(n) = \sum_{i=1}^n 2^{-i} = 1 - 2^{-n}$  is not a rational function, but  $\lim_{n \rightarrow \infty} S(n) = 1 \in \mathbb{Q}$ .

**Theorem 2.5.** *Let  $K$  be a subfield of  $\mathbb{C}$ , and let  $f(n)$  be a function in one positive integer variable  $n$ . Assume that  $\lim_{n \rightarrow \infty} f(n)$  is a transcendental number over  $K$ . Then  $f(n)$  is a transcendental function in  $n$  over  $K$ .*

*Proof.* Arguing by contradiction, suppose that  $f(n)$  is a algebraic function over  $K$ . Then there exist polynomials in  $n$  with coefficients in over  $K$ :  $q_0(n), \dots, q_k(n) \in K[n]$  with  $q_k(n) \neq 0$  such that

$$q_k(n)[f(n)]^k + q_{k-1}(n)[f(n)]^{k-1} + \dots + q_1(n)[f(n)] + q_0(n) = 0.$$

Set  $d = \max\{\deg q_0(n), \dots, \deg q_k(n)\}$ . It is easily seen that  $\lim_{n \rightarrow \infty} \frac{q_i(n)}{n^d} \in K$  for all  $0 \leq i \leq k$ . Set  $\lim_{n \rightarrow \infty} f(n) = \alpha$  and  $\lim_{n \rightarrow \infty} \frac{q_i(n)}{n^d} = c_i$  for all  $0 \leq i \leq k$ . Then  $\alpha$  is a transcendental number over  $K$  and  $c_0, \dots, c_k \in K$  are not all zero. Since

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left( \left[ \frac{q_k(n)}{n^d} \right] [f(n)]^k + \left[ \frac{q_{k-1}(n)}{n^d} \right] [f(n)]^{k-1} + \dots + \left[ \frac{q_1(n)}{n^d} \right] [f(n)] + \left[ \frac{q_0(n)}{n^d} \right] \right) \\ &= \lim_{n \rightarrow \infty} \left[ \frac{q_k(n)}{n^d} \right] \lim_{n \rightarrow \infty} [f(n)]^k + \lim_{n \rightarrow \infty} \left[ \frac{q_{k-1}(n)}{n^d} \right] \lim_{n \rightarrow \infty} [f(n)]^{k-1} \\ &\quad + \dots + \lim_{n \rightarrow \infty} \left[ \frac{q_1(n)}{n^d} \right] \lim_{n \rightarrow \infty} [f(n)] + \lim_{n \rightarrow \infty} \left[ \frac{q_0(n)}{n^d} \right] \\ &= c_k \alpha^k + c_{k-1} \alpha^{k-1} + \dots + c_1 \alpha + c_0, \end{aligned}$$

we obtain  $c_k\alpha^k + c_{k-1}\alpha^{k-1} + \cdots + c_1\alpha + c_0 = 0$  for  $c_0, \dots, c_k \in K$ , not all zero; i.e.,  $\alpha$  is algebraic over  $K$ . This contradiction concludes the proof.  $\square$

Remember that  $\overline{\mathbb{Q}}$  is algebraic over  $\mathbb{Q}$ ,  $\alpha \in \mathbb{C}$  is algebraic over  $\overline{\mathbb{Q}}$  if and only if  $\alpha$  is an algebraic number (see e.g. [6, 8]). Consequently, we immediately have the following consequence by Theorem 2.5.

**Corollary 2.6.** *Let  $f(n)$  be a function in one positive integer variable  $n$ . Assume that  $\lim_{n \rightarrow \infty} f(n)$  is a transcendental number. Then  $f(n)$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$ .*

**Remark 2.7.**  $F(n) = 1 + 3^{-n}$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$  which tends to 1 (an algebraic number). Hence the reverse of Corollary 2.6 does not hold.

Now, assume that  $(u_i)_{i \geq 1}$  is a rational number sequence and  $S(n) = \sum_{i=1}^n u_i$ . Then  $S(n)$  is a function in  $n$ . Hence as an immediate consequence of Corollary 2.6 we obtain the following result.

**Corollary 2.8.** *Let  $(u_i)_{i \geq 1}$  be a rational number sequence. Assume that  $\lim_{n \rightarrow \infty} S(n)$  is a transcendental number. Then  $S(n) = \sum_{i=1}^n u_i$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$ .*

**Remark 2.9.** The reverse of Corollary 2.8 does not hold. Indeed, now we choose  $(u_i = 3^{-i})_{i \geq 1}$ , then  $S(n) = \sum_{i=1}^n 3^{-i} = \frac{1}{2}(1 - 3^{-n})$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$ , however  $\lim_{n \rightarrow \infty} S(n) = \frac{1}{2}$  is an algebraic number.

### 3 Some Applications

In this section, we give some applications of Section 2 for partial sums of famous rational number sequences in mathematical analysis with relevance to number theory.

**Example 3.1.** Let  $a$  and  $b$  be non-negative integers with  $a \neq 0$ . Then

$$S(n) = \sum_{j=1}^n \frac{1}{aj + b}$$

is not a rational function in  $n$  over  $\mathbb{Q}$ .

*Proof.* Set  $c = \max\{a, b\}$ . Since

$$S(n) = \sum_{j=1}^n \frac{1}{aj + b} \geq \sum_{j=1}^n \frac{1}{cj + c} = \frac{1}{c} \sum_{j=1}^n \frac{1}{j + 1}$$

and  $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{1}{j+1} = \infty$  (see e.g. [5]), it follows that  $\lim_{n \rightarrow \infty} S(n) = \infty$ . Now, we will prove that

$$\sum_{j=1}^n \frac{1}{j} < \sqrt{2n+1}$$

by induction on  $n$ . The result is true for  $n = 1$  since  $1 < \sqrt{3}$ . Next assume that  $n > 1$ . By the inductive assumption we have

$$\sum_{j=1}^n \frac{1}{j} = \sum_{j=1}^{n-1} \frac{1}{j} + \frac{1}{n} < \sqrt{2(n-1)+1} + \frac{1}{n}.$$

Since  $\sqrt{2n+1} - \sqrt{2(n-1)+1} = \frac{2}{\sqrt{2n+1} + \sqrt{2(n-1)+1}}$ , it follows that

$$\sqrt{2(n-1)+1} + \frac{1}{n} < \sqrt{2n+1}$$

if and only if  $2n > \sqrt{2n+1} + \sqrt{2(n-1)+1}$ . This is equivalent to  $4n^3(n-2)+1 > 0$ . Remember that  $n \geq 2$ ,  $4n^3(n-2)+1 > 0$ . So  $2n > \sqrt{2n+1} + \sqrt{2(n-1)+1}$  for all  $n \geq 2$ . Induction is complete. We obtain  $S(n) \leq \sum_{j=1}^n \frac{1}{j} < \sqrt{2n+1}$ . From this it follows that  $\lim_{n \rightarrow \infty} \frac{S(n)}{n} = 0$ . Thus,  $S(n)$  is not a rational function over  $\mathbb{Q}$  by Corollary 2.3(i).  $\square$

**Note 1:** In mathematics, the  $n$ -th harmonic number is the sum of the reciprocals of the first  $n$  natural numbers:  $H_n = \sum_{j=1}^n \frac{1}{j}$ . Example 3.1 showed that  $H_n$  is not a rational function in  $n$  over  $\mathbb{Q}$ . Euler found in 1734 that  $\lim_{n \rightarrow \infty} [H_n - \ln n]$  is a constant called the Euler's constant and one denoted by  $\gamma$  this constant (see e.g. [3, 10, 12]).  $\gamma = 0.577215665\dots$  has not yet been proven to be transcendental or even irrational. Sondow gave in 2002 [11] criteria for irrationality of  $\gamma$ . Now, we need to talk more about  $H_n$ . Denote by  $\psi(x)$  the digamma function, that is, the logarithmic derivative of Euler's  $\Gamma$ -function. Then we have  $H_n = \psi(n+1) + \gamma$  (see e.g. [9]). By Murty and Saradha in 2007 [9, Theorem 1],  $\psi(x) + \gamma$  takes transcendental values at infinitely many  $x$  rational. Consequently,  $\psi(x) + \gamma$  is a transcendental function over  $\overline{\mathbb{Q}}$ . Hence we would like to give a conjecture that " $H_n = \psi(n+1) + \gamma$  is a transcendental function over  $\overline{\mathbb{Q}}$ ".

**Example 3.2.** Let  $(u_n)_{n \geq 1}$  be a sequence defined by  $u_1 = 1$  and

$$u_{i+1} = \frac{u_i + \frac{2}{u_i}}{2}$$

for all  $i \geq 1$ . Then the function  $F(n) = u_n$  in  $n$  is not a rational function over  $\mathbb{Q}$ .

*Proof.* Since  $u_{i+1} = \frac{u_i + \frac{2}{u_i}}{2} \geq \sqrt{u_i \cdot \frac{2}{u_i}} = \sqrt{2}$  for all  $i \geq 1$ , it follows that

$$u_{i+1} = \frac{u_i + \frac{2}{u_i}}{2} \leq \frac{u_i + \sqrt{2}}{2} \leq u_i$$

for all  $i \geq 2$ . So  $(u_n)_{n \geq 1}$  is convergent. Hence there exists  $\lim_{n \rightarrow \infty} u_n$ . Set

$$\lim_{n \rightarrow \infty} u_n = x.$$

Then  $x = \frac{x + \frac{2}{x}}{2}$  since  $u_{i+1} = \frac{u_i + \frac{2}{u_i}}{2}$ . Consequently,  $x = \sqrt{2}$ . Since  $\lim_{n \rightarrow \infty} u_n = \sqrt{2}$  is an irrational number, the function  $F(n) = u_n$  in  $n$  is not a rational function over  $\mathbb{Q}$  by Theorem 2.1(iii). □

**Example 3.3.**  $F(n) = \sum_{j=1}^n \frac{(-1)^{j-1}}{j}$ ;  $L(n) = \sum_{j=1}^n \frac{(-1)^{j-1}}{2j-1}$ ;  $S(n) = \sum_{j=1}^n \frac{1}{j!}$  are transcendental functions in  $n$  over  $\overline{\mathbb{Q}}$ .

*Proof.* Remember that  $\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} = \ln 2$  (see e.g. [5]). So  $\lim_{n \rightarrow \infty} F(n) = \ln 2$ . Since  $2 \in \mathbb{Q} \setminus \{0, 1\}$ ,  $\ln 2$  is transcendental by the Hermite-Lindemann theorem (see e.g. [6, Appendix 1]). Hence  $F(n)$  is transcendental over  $\overline{\mathbb{Q}}$  by Corollary 2.8. By the Leibnitz formula for  $\pi$ , we have

$$\lim_{n \rightarrow \infty} L(n) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} = \frac{\pi}{4}$$

(see e.g. [2, 4, 5, 13]). Since  $\pi$  is a transcendental number by the Lindemann theorem (see e.g. [1, 6, 7]),  $L(n)$  is a transcendental function over  $\overline{\mathbb{Q}}$  by Corollary 2.8. Since

$$1 + \lim_{n \rightarrow \infty} S(n) = \sum_{j=0}^{\infty} \frac{1}{j!} = e$$

(see e.g. [5]) is a transcendental number by the Hermite theorem (see e.g. [1, 6, 7]),

$$S(n) = \sum_{j=1}^n \frac{1}{j!}$$

is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$  by Corollary 2.8. □

**Example 3.4.**  $U(n) = \sum_{j=1}^n \frac{1}{j^2}$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$ .

*Proof.* Recall that the Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e., the precise sum of the infinite series:

$$\sum_{j=1}^{\infty} \frac{1}{j^2}.$$

This problem is a famous problem in mathematical analysis with relevance to number theory, first posed by Pietro Mengoli in 1644 and solved by Leonhard Euler in 1735 that  $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$  (see e.g. [3, 12]). So

$$\lim_{n \rightarrow \infty} U(n) = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Since  $\pi$  is a transcendental number by the Lindemann theorem (see e.g. [1, 6, 7]),  $\frac{\pi^2}{6}$  is also a transcendental number. Hence  $U(n) = \sum_{j=1}^n \frac{1}{j^2}$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$  by Corollary 2.8.  $\square$

**Note 2:** Recall that  $\psi(x)$  is the digamma function, that is, the logarithmic derivative of Euler's  $\Gamma$ -function. Then it is known that  $\sum_{j=1}^n \frac{1}{j^2} = \frac{\pi^2}{6} - \psi'(n+1)$  (see e.g. [9]). Hence

$$\lim_{n \rightarrow \infty} \left[ \frac{\pi^2}{6} - \psi'(n) \right] = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

So  $\psi'(n) - \frac{\pi^2}{6}$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$  by Corollary 2.6. Now, since  $\psi'(n+1) = \frac{\pi^2}{6} - \sum_{j=1}^n \frac{1}{j^2}$ , it follows that  $\psi'(k)$  takes transcendental values for all  $k$  positive integer. Hence  $\psi'(n)$  is a transcendental function in  $n$  over  $\overline{\mathbb{Q}}$ .

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